Almost Gorenstein rings

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Based on the works jointly with

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September 10, 2016

Introduction

Question 1.1

Why are there so many Cohen-Macaulay rings which are not Gorenstein?

Let R be a Noetherian ring. Then

$$R \text{ is a Gorenstein ring} \underset{def}{\Longleftrightarrow} \mathrm{id}_{R_\mathfrak{p}} R_\mathfrak{p} < \infty \quad \text{for } \forall \mathfrak{p} \in \operatorname{Spec} R.$$

Example 1.2

Let $S = k[X_{ij} \mid 1 \le i \le m, 1 \le j \le n\,]$ $(2 \le m \le n)$ be the polynomial ring over a field k and put

$$R = S/\mathrm{I}_t(X)$$

where $2 \le t \le m$, $\mathrm{I}_t(X)$ is the ideal of S generated by $t \times t$-minors of $X = (X_{ij})$.

Then R is a Gorenstein ring if and only if $m = n$.

Aim of my talk

Find a new class of Cohen-Macaulay rings which may not be Gorenstein, but sufficiently good next to Gorenstein rings.

Introduction

History of almost Gorenstein rings

- [Barucci-Fröberg, 1997]

  $\cdots$ one-dimensional analytically unramified local rings

- [Goto-Matsuoka-Phuong, 2013]

  $\cdots$ one-dimensional Cohen-Macaulay local rings

- [Goto-Takahashi-T, 2015]

  $\cdots$ higher-dimensional Cohen-Macaulay local/graded rings

Survey on one-dimensional almost Gorenstein local rings

Setting 2.1

- $(R, \mathfrak{m})$ a Cohen-Macaulay local ring with $\dim R = 1$
- $|R/\mathfrak{m}| = \infty$
- $\exists\ \mathrm{K}_R$ the canonical module of R
- $\exists\ I \subsetneq R$ an ideal of R such that $I \cong \mathrm{K}_R$

Therefore $\exists\ \mathrm{e}_0(I) > 0,\ \mathrm{e}_1(I) \in \mathbb{Z}$ such that

$$\ell_R(R/I^{n+1}) = \mathrm{e}_0(I)\binom{n+1}{1} - \mathrm{e}_1(I)$$

for $\forall n \gg 0$.

Set $\mathrm{r}(R) = \ell_R(\mathrm{Ext}^1_R(R/\mathfrak{m}, R))$.

Definition 2.2 (Goto-Matsuoka-Phuong)

We say that R is *an almost Gorenstein local ring*, if $\mathrm{e}_1(I) \leq \mathrm{r}(R)$.

Suppose that I contains a parameter ideal $Q = (a)$ as a reduction, i.e.

$$I^{r+1} = QI^r \quad \text{for} \quad \exists\, r \geq 0.$$

We set

$$K = \frac{I}{a} = \left\{ \frac{x}{a} \mid x \in I \right\} \subseteq \mathrm{Q}(R).$$

Then K is a fractional ideal of R such that

$$R \subseteq K \subseteq \overline{R} \quad \text{and} \quad K \cong \mathrm{K}_R\,.$$

Theorem 2.3 (Goto-Matsuoka-Phuong)

R is an almost Gorenstein local ring $\Longleftrightarrow$ $\mathfrak{m}K \subseteq R$ (*i.e.* $\mathfrak{m}I = \mathfrak{m}Q$)

Example 2.4

Let k be an infinite field.

(1) $k[[t^3, t^4, t^5]]$

(2) $k[[t^a, t^{a+1}, \dots, t^{2a-3}, t^{2a-1}]]$ $(a \geq 4)$

(3) $k[[X, Y, Z]]/(X, Y) \cap (Y, Z) \cap (Z, X)$

(4) $k[[X, Y, Z, U, V, W]]/I$, where

$$I = (X^3 - Z^2, Y^2 - ZX) + (U, V, W)^2 + (YU - XV, ZU - XW, ZU - YV, ZV - YW, X^2U - ZW)$$

Almost Gorenstein local rings of higher dimension

Setting 3.1

- $(R, \mathfrak{m})$ a Cohen-Macaulay local ring with $d = \dim R$
- $|R/\mathfrak{m}| = \infty$
- $\exists$ K_R the canonical module of R

Definition 3.2

We say that R is *an almost Gorenstein local ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$.

Look at an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules. If $C \neq (0)$, then C is Cohen-Macaulay and $\dim_R C = d-1$.

Set $\overline{R} = R/[(0) :_R C]$.

Then $\exists\ f_1, f_2, \dots, f_{d-1} \in \mathfrak{m}$ s.t. $(f_1, f_2, \dots, f_{d-1})\overline{R}$ forms a minimal reduction of $\overline{\mathfrak{m}} = \mathfrak{m}\overline{R}$. Therefore

$$\mathrm{e}^0_{\mathfrak{m}}(C) = \mathrm{e}^0_{\overline{\mathfrak{m}}}(C) = \ell_R(C/(f_1, f_2, \dots, f_{d-1})C) \geq \ell_R(C/\mathfrak{m}C) = \mu_R(C).$$

Thus

$$\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C) \Longleftrightarrow \mathfrak{m}C = (f_1, f_2, \dots, f_{d-1})C.$$

Hence C is a *maximally generated maximal Cohen-Macaulay $\overline{R}$-module* in the sense of B. Ulrich, which is called *an Ulrich R-module*.

Definition 3.3

We say that R is *an almost Gorenstein local ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that either $C = (0)$ or $C \neq (0)$ and C is an Ulrich R-module.

Remark 3.4

Suppose that $d = 1$. Then TFAE.

(1) R is almost Gorenstein in the sense of Definition 3.3.

(2) R is almost Gorenstein in the sense of [GMP, Definition 3.1].

Definition 3.3

We say that R is *an almost Gorenstein local ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that either $C = (0)$ or $C \neq (0)$ and C is an Ulrich R-module.

Remark 3.4

Suppose that $d = 1$. Then TFAE.

(1) R is almost Gorenstein in the sense of Definition 3.3.

(2) R is almost Gorenstein in the sense of [GMP, Definition 3.1].

Theorem 3.5 (NZD characterization)

(1) *If R is a non-Gorenstein almost Gorenstein local ring of dimension $d > 1$, then so is $R/(f)$ for genaral NZD $f \in \mathfrak{m} \setminus \mathfrak{m}^2$.*

(2) *Let $f \in \mathfrak{m}$ be a NZD on R. If $R/(f)$ is an almost Gorenstein local ring, then so is R. When this is the case, $f \notin \mathfrak{m}^2$, if R is not Gorenstein.*

Corollary 3.6

Suppose that $d > 0$. If $R/(f)$ is an almost Gorenstein local ring for every NZD $f \in \mathfrak{m}$, then R is Gorenstein.

We set $\mathrm{r}(R) = \ell_R(\mathrm{Ext}_R^d(R/\mathfrak{m}, R))$.

Example 3.7

Let $U = k[[X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n]]$ $(n \geq 2)$ be the formal power series ring over an infinite field k and put

$$R = U/\mathrm{I}_2(\mathbb{M}),$$

where $\mathrm{I}_2(\mathbb{M})$ denotes the ideal of U generated by 2×2 minors of the matrix

$$\mathbb{M} = \begin{pmatrix} X_1 & X_2 & \cdots & X_n \\ Y_1 & Y_2 & \cdots & Y_n \end{pmatrix}.$$

Then R is almost Gorenstein with $\dim R = n + 1$ and $\mathrm{r}(R) = n - 1$.

Proof of Example 3.7.

Notice that

- $\{X_i - Y_{i-1}\}_{1 \le i \le n}$ (here $Y_0 = Y_n$) forms a regular sequence on R
- $R/(X_i - Y_{i-1} \mid 1 \le i \le n)R \;\cong\; k[[X_1, X_2, \ldots, X_n]]/\mathrm{I}_2(\mathbb{N}) = S$

  where $\mathbb{N} = \begin{pmatrix} X_1 & X_2 & \cdots & X_{n-1} & X_n \\ X_2 & X_3 & \cdots & X_n & X_1 \end{pmatrix}$.

Then

- S is Cohen-Macaulay with $\dim S = 1$
- $\mathfrak{n}^2 = x_1\mathfrak{n}$ and $\mathrm{K}_S \cong (x_1, x_2, \ldots, x_{n-1})$

where $\mathfrak{n}$ is the maximal ideal of S, x_i is the image of X_i in S.

Hence S is an almost Gorenstein local ring, since $\mathfrak{n}(x_1, x_2, \ldots, x_{n-1}) \subseteq (x_1)$.

Thus R is almost Gorenstein. □

Theorem 3.8

Let $(S, \mathfrak{n})$ be a Noetherian local ring, $\varphi : R \to S$ a flat local homomorphism. Suppose that $S/\mathfrak{m}S$ is a RLR. Then TFAE.

(1) *R is an almost Gorenstein local ring.*

(2) *S is an almost Gorenstein local ring.*

Therefore

- R is almost Gorenstein $\Longleftrightarrow$ $R[[X_1, X_2, \ldots, X_n]]$ $(n \geq 1)$ is almost Gorenstein.
- R is almost Gorenstein $\Longleftrightarrow$ $\widehat{R}$ is almost Gorenstein.

The following is a generalization of [GMP, Theorem 6.5].

Theorem 3.9

Suppose that $d > 0$. Let $\mathfrak{p} \in \operatorname{Spec} R$ and assume that $R/\mathfrak{p}$ is a RLR of dimension $d-1$. Then TFAE.

(1) *$A = R \ltimes \mathfrak{p}$ is an almost Gorenstein local ring.*

(2) *R is an almost Gorenstein local ring.*

Example 3.10

Let k be an infinite field.

We consider $A = k[[X, Y, Z, U, V, W]]/I$, where

$I = (X^3 - Z^2, Y^2 - ZX) + (U, V, W)^2 + (YU - XV, ZU - XW, ZU - YV, ZV - YW, X^2U - ZW)$.

Then

$$A \cong k[[t^4, t^5, t^6]] \ltimes (t^4, t^5, t^6)$$

and hence A is an almost Gorenstein local ring.

Theorem 3.11

Suppose that $d > 0$ and $\mathrm{Q}(R)$ is a Gorenstein ring. Let I ($\subsetneq R$) be an ideal of R such that $I \cong \mathrm{K}_R$. Then TFAE.

(1) *R is an almost Gorenstein local ring.*

(2) *R contains a parameter ideal $Q = (f_1, f_2, \ldots, f_d)$ such that $f_1 \in I$ and $\mathfrak{m}(I+Q) = \mathfrak{m}Q$.*

When this is the case, if $d \geq 2$ and R is not a Gorenstein ring, we have the following, where $J = I + Q$.

(a) $\mathrm{red}_Q(J) = 2$.

(b) $\ell_R(R/J^{n+1}) = \ell_R(R/Q) \cdot \binom{n+d}{d} - \mathrm{r}(R) \cdot \binom{n+d-1}{d-1} + \binom{n+d-2}{d-2}$ *for* $\forall n \geq 0$.

Hence $\mathrm{e}_1(J) = \mathrm{r}(R)$.

(c) *Let $G = \mathrm{gr}_J(R)$. Then $f_2, f_3, \ldots, f_d$ is a super-regular sequence with respect to J and* depth $G = d - 1$.

Theorem 3.12

Let $(R, \mathfrak{m})$ *be a Cohen-Macaulay complete local ring with* $\dim R = 1$ *and assume that* $R/\mathfrak{m}$ *is algebraically closed of characteristic* 0.
Suppose that R *has finite CM representation type. Then* R *is an almost Gorenstein local ring.*

Theorem 3.13 (Goto)

Suppose that R is a non-Gorenstein almost Gorenstein local ring with $\dim R \geq 1$. Let M be a finitely generated R-module. If

$$\mathrm{Ext}_R^i(M, R) = (0)$$

for $\forall i \gg 0$, then $\mathrm{pd}_R M < \infty$.

Corollary 3.14

Suppose that R is an almost Gorenstein local ring with $\dim R \geq 1$. If R is not a Gorenstein ring, then R is G-regular in the sense of [7], i.e.

$$\mathrm{Gdim}_R M = \mathrm{pd}_R M$$

for every finitely generated R-module M.

Semi-Gorenstein local rings

In this section we maintain Setting 3.1.

Let $\mathcal{F} = \{I_n\}_{n\in\mathbb{Z}}$ be a filtration of ideals of R s.t. $I_0 = R$, $I_1 \neq R$.

We consider the R-algebras

$$\mathcal{R} = \sum_{n\geq 0} I_n t^n \subseteq R[t], \quad \mathcal{R}' = \sum_{n\in\mathbb{Z}} I_n t^n \subseteq R[t, t^{-1}], \quad \text{and} \quad G = \mathcal{R}'/t^{-1}\mathcal{R}'$$

associated to $\mathcal{F}$, where t is an indeterminate.

Let N denote the graded maximal ideal of $\mathcal{R}'$.

Theorem 4.1

Suppose that $\mathcal{R}$ is a Noetherian ring. If G_N is an almost Gorenstein local ring and $\mathrm{r}(G_N) \le 2$, then R is almost Gorenstein.

Proof.

We may assume $\mathrm{r}(G_N) = 2$. Since $\mathcal{R}'_N$ is an almost Gorenstein local ring with $\mathrm{r}(\mathcal{R}'_N) = 2$, we have

$$0 \to \mathcal{R}'_N \to \mathrm{K}_{(\mathcal{R}'_N)} \to C \to 0$$

where $C \cong$ a RLR of dim d.

Let $\mathfrak{p} = \mathfrak{m}R[t, t^{-1}]$ and set $P = \mathfrak{p} \cap \mathcal{R}'$. Then $P \subseteq N$, so that $R[t, t^{-1}]_{\mathfrak{p}}$ is an almost Gorenstein local ring, because

$$R[t, t^{-1}]_{\mathfrak{p}} = \mathcal{R}'_P = (\mathcal{R}'_N)_{P\mathcal{R}'_N}.$$

Hence R is an almost Gorenstein local ring, since $R \to R[t, t^{-1}] \to R[t, t^{-1}]_{\mathfrak{p}}$ is a flat homomorphism. $\square$

Example 4.2 (Barucci-Dobbs-Fontana)

Let $R = k[[x^4, x^6 + x^7, x^{10}]] \subseteq V$, where $V = k[[x]]$ denotes the formal power series ring over an infinite field k of $\mathrm{ch}\, k \neq 2$.

Let $H = \{v(a) \mid 0 \neq a \in R\}$ be the value semigroup of R.

We consider the filtration $\mathcal{F} = \{(xV)^n \cap R\}_{n \in \mathbb{Z}}$ of ideals of R. We then have the following.

(1) $H = \langle 4, 6, 11, 13 \rangle$.

(2) $G \cong k[x^4, x^6, x^{11}, x^{13}]$ $(\subseteq k[x])$ and G_N is an almost Gorenstein local ring with $\mathrm{r}(G_N) = 3$.

(3) R is NOT an almost Gorenstein local ring and $\mathrm{r}(R) = 2$.

Therefore $(\mathcal{R}'_N)_{P\mathcal{R}'_N}$ is NOT an almost Gorenstein local ring.

Definition 4.3

We say that R is *a semi-Gorenstein local ring*, if R is an almost Gorenstein local ring which possesses an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

such that either $C = (0)$, or C is an Ulrich R-module and $C = \oplus_{i=1}^{\ell} C_i$ for some cyclic R-submodule C_i of C.

Therefore, if $C \neq (0)$, then

$$C_i \cong R/\mathfrak{p}_i \quad \text{for } \exists\, \mathfrak{p}_i \in \operatorname{Spec} R$$

such that $R/\mathfrak{p}_i$ is a RLR of dimension $d-1$.

Notice that

- almost Gorenstein local ring with $\dim R = 1$
- almost Gorenstein local ring with $\mathrm{r}(R) \leq 2$

are semi-Gorenstein.

Proposition 4.4

Let R be a semi-Gorenstein local ring. Then $R_\mathfrak{p}$ is semi-Gorenstein for $\forall \mathfrak{p} \in \operatorname{Spec} R$.

Therefore, if $C \neq (0)$, then

$$C_i \cong R/\mathfrak{p}_i \quad \text{for } \exists\ \mathfrak{p}_i \in \operatorname{Spec} R$$

such that $R/\mathfrak{p}_i$ is a RLR of dimension $d-1$.

Notice that

- almost Gorenstein local ring with $\dim R = 1$
- almost Gorenstein local ring with $\mathrm{r}(R) \leq 2$

are semi-Gorenstein.

Proposition 4.4

Let R be a semi-Gorenstein local ring. Then $R_{\mathfrak{p}}$ is semi-Gorenstein for $\forall \mathfrak{p} \in \operatorname{Spec} R$.

Theorem 4.5

Let $(S, \mathfrak{n})$ be a RLR, $\mathfrak{a} \subsetneq S$ an ideal of S with $n = \mathrm{ht}_S\, \mathfrak{a}$. Let $R = S/\mathfrak{a}$. Then TFAE.

(1) *R is a semi-Gorenstein local ring, but not Gorenstein.*

(2) *R is Cohen-Macaulay, $n \geq 2$, $r = \mathrm{r}(R) \geq 2$, and R has a minimal S-free resolution of the form:*

$$0 \to F_n = S^r \xrightarrow{\mathbb{M}} F_{n-1} = S^q \to F_{n-2} \to \cdots \to F_1 \to F_0 = S \to R \to 0$$

where

$${}^t\mathbb{M} = \begin{pmatrix} y_{21}y_{22}\cdots y_{2\ell} & y_{31}y_{32}\cdots y_{3\ell} & \cdots & y_{r1}y_{r2}\cdots y_{r\ell} & z_1 z_2 \cdots z_m \\ x_{21}x_{22}\cdots x_{2\ell} & 0 & 0 & 0 & 0 \\ 0 & x_{31}x_{32}\cdots x_{3\ell} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & x_{r1}x_{r2}\cdots x_{r\ell} & 0 \end{pmatrix},$$

$\ell = n+1$, $q \geq (r-1)\ell$, $m = q - (r-1)\ell$, and $x_{i1}, x_{i2}, \ldots, x_{i\ell}$ is a part of a regular system of parameters of S for $2 \leq \forall i \leq r$.

When this is the case

$$\mathfrak{a} = (z_1, z_2, \ldots, z_m) + \sum_{i=2}^{r} \mathrm{I}_2 \begin{pmatrix} y_{i1} & y_{i2} & \cdots & y_{i\ell} \\ x_{i1} & y_{i2} & \cdots & x_{i\ell} \end{pmatrix}.$$

Example 4.6

Let $\varphi : S = k[[X, Y, Z, W]] \longrightarrow R = k[[t^5, t^6, t^7, t^9]]$ be the k-algebra map defined by

$$\varphi(X) = t^5,\ \varphi(Y) = t^6,\ \varphi(Z) = t^7 \text{ and } \varphi(W) = t^9.$$

Then

$$0 \to S^2 \xrightarrow{\mathbb{M}} S^6 \to S^5 \to S \to R \to 0,$$

where

$${}^t\mathbb{M} = \begin{pmatrix} W & X^2 & XY & YZ & Y^2-XZ & Z^2-XW \\ X & Y & Z & W & 0 & 0 \end{pmatrix}.$$

Hence R is semi-Gorenstein with $\mathrm{r}(R) = 2$ and

$$\operatorname{Ker}\varphi = (Y^2 - XZ, Z^2 - XW) + \mathrm{I}_2\begin{pmatrix} W & X^2 & XY & YZ \\ X & Y & Z & W \end{pmatrix}.$$

Almost Gorenstein graded rings

Setting 5.1

- $R = \bigoplus_{n \geq 0} R_n$ a Cohen-Macaulay graded ring with $d = \dim R$
- $(R_0, \mathfrak{m})$ a Noetherian local ring
- $|R_0/\mathfrak{m}| = \infty$
- $\exists\ \mathrm{K}_R$ the graded canonical module of R
- $M = \mathfrak{m}R + R_+$
- $a = \mathrm{a}(R) := -\min\{n \in \mathbb{Z} \mid [\mathrm{K}_R]_n \neq (0)\}$

Definition 5.2

We say that R is *an almost Gorenstein graded ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R(-a) \to C \to 0$$

of graded R-modules such that $\mu_R(C) = \mathrm{e}^0_M(C)$.

Notice that

- R is an almost Gorenstein graded ring $\Longrightarrow R_M$ is an almost Gorenstein local ring.

Example 5.3

Let $U = k[s,t]$ be the polynomial ring over an infinite field k and look at the subring $R = k[s, s^3t, s^3t^2, s^3t^3] \subseteq U$.

Let $S = k[X,Y,Z,W]$ be the weighted polynomial ring s.t.

$$\deg X = 1, \quad \deg Y = 4, \quad \deg Z = 5, \quad \text{and} \quad \deg W = 6.$$

Let $\psi : S \to R$ be the k-algebra map defined by

$$\psi(X) = s, \quad \psi(Y) = s^3t, \quad \psi(Z) = s^3t^2, \quad \text{and} \quad \psi(W) = s^3t^3.$$

Then $\operatorname{Ker}\psi = \mathrm{I}_2\begin{pmatrix} X^3 & Y & Z \\ Y & Z & W \end{pmatrix}$ and R has a graded minimal S-free resolution

$$0 \to S(-13) \oplus S(-14) \xrightarrow{\begin{pmatrix} X^3 & Y \\ Y & Z \\ Z & W \end{pmatrix}} S(-10) \oplus S(-9) \oplus S(-8) \xrightarrow{(\Delta_1\ \Delta_2\ \Delta_3)} S \xrightarrow{\psi} R \to 0$$

where $\Delta_1 = Z^2 - YW$, $\Delta_2 = X^3W - YZ$, and $\Delta_3 = Y^2 - X^3Z$.

Example

Therefore, because $\mathrm{K}_S \cong S(-16)$, we get

$$S(-6) \oplus S(-7) \oplus S(-8) \xrightarrow{\begin{pmatrix} X^3 & Y & Z \\ Y & Z & W \end{pmatrix}} S(-3) \oplus S(-2) \xrightarrow{\varepsilon} \mathrm{K}_R \to 0. \qquad (\sharp)$$

Hence $\mathrm{a}(R) = -2$. Let $\xi = \varepsilon(\binom{1}{0}) \in [\mathrm{K}_R]_3$ and we have

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R(3) \to S/(Y, Z, W)(1) \to 0$$

where $\varphi(1) = \xi$. Hence R_M is a semi-Gorenstein local ring.

On the other hand, by $(\sharp)$ we get $[\mathrm{K}_R]_2 = k\eta \neq (0)$, where $\eta = \varepsilon(\binom{0}{1})$.

Hence if R is an almost Gorenstein graded ring, we must have

$$\mu_R(\mathrm{K}_R / R\eta) = \mathrm{e}^0_M(\mathrm{K}_R / R\eta)$$

which is impossible, because $\mathrm{K}_R / R\eta \cong [S/(X^3, Y, Z)](-3)$.

Theorem 5.4

Let $R = k[R_1]$ be a Cohen-Macaulay homogeneous ring with $d = \dim R \geq 1$. Suppose that $|k| = \infty$ and R is not a Gorenstein ring. Then TFAE.

(1) *R is an almost Gorenstein graded ring and level.*

(2) *$\mathrm{Q}(R)$ is a Gorenstein ring and $\mathrm{a}(R) = 1 - d$.*

Example 5.5

Let $S = k[X_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\,]$ $(2 \leq m \leq n)$ be the polynomial ring over an infinite field k and put

$$R = S/\mathrm{I}_t(X)$$

where $2 \leq t \leq m$, $X = [X_{ij}]$.

Then R is an almost Gorenstein graded ring if and only if either $m = n$, or $m \neq n$ and $t = m = 2$.

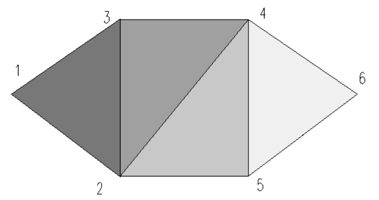
Example 5.6

Let $R = k[X_1, X_2, \ldots, X_d]$ $(d \geq 1)$ be a polynomial ring over an infinite field k. Let $n \geq 1$ be an integer.

- $R^{(n)} = k[R_n]$ is an almost Gorenstein graded ring, if $d \leq 2$.
- Suppose that $d \geq 3$. Then $R^{(n)}$ is an almost Gorenstein graded ring if and only if either $n \mid d$, or $d = 3$ and $n = 2$.

Example 5.7

Look at the simplicial complex Δ :



Then $R = k[\Delta]$ is an almost Gorenstein graded ring of dimension 3, provided $|k| = \infty$.

Theorem 5.8 (Goto-Iai)

Let R be a Gorenstein local ring, $I \subsetneq R$ an ideal of R. If $G = \mathrm{gr}_I(R)$ is an almost Gorenstein graded ring, then G is Gorenstein.

Theorem 5.9

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $|R/\mathfrak{m}| = \infty$, $\exists\, \mathrm{K}_R$. Let I be an $\mathfrak{m}$-primary ideal of R.
If $G = \mathrm{gr}_I(R)$ is an almost Gorenstein graded ring and $\mathrm{r}(G) = \mathrm{r}(R)$, then R is an almost Gorenstein local ring.

Two-dimensional rational singularities

Setting 6.1

- $(R, \mathfrak{m})$ a Cohen-Macaulay local ring with $d = \dim R$
- $|R/\mathfrak{m}| = \infty$
- $\exists$ K_R the canonical module of R
- $v(R) = \mu_R(\mathfrak{m})$, $\mathrm{e}(R) = \mathrm{e}^0_{\mathfrak{m}}(R)$
- $G = \mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$

Theorem 6.2

(1) *Suppose that R is an almost Gorenstein local ring and $v(R) = \mathrm{e}(R) + d - 1$. Then G is an almost Gorenstein graded ring and level.*

(2) *Suppose that G is an almost Gorenstein graded ring and level. Then R is an almost Gorenstein local ring.*

Corollary 6.3

Suppose that $v(R) = e(R) + d - 1$. *Then TFAE.*

(1) *R is an almost Gorenstein local ring.*

(2) *G is an almost Gorenstein graded ring.*

(3) $Q(G)$ *is a Gorenstein ring.*

Corollary 6.4

Suppose that $v(R) = e(R) + d - 1$ *and R is a normal ring. If* $\mathfrak{m}$ *is a normal ideal, then R is an almost Gorenstein local ring.*

Corollary 6.5

Every two-dimensional rational singularity is an almost Gorenstein local ring.

Corollary 6.6

Every two-dimensional Cohen-Macaulay complete local ring R of finite CM representation type is an almost Gorenstein local ring, provided R contains a field of characteristic 0.

Almost Gorenstein Rees algebras

Setting 7.1

- $(R, \mathfrak{m})$ a Gorenstein local ring with $\dim R = 2$
- $|R/\mathfrak{m}| = \infty$
- $\sqrt{I} = \mathfrak{m}$
- I contains a parameter ideal Q s.t. $I^2 = QI$
- $J = Q : I$
- $\mathcal{R} = \mathcal{R}(I) := R[It] \subseteq R[t]$ the Rees algebra of I
- $M = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$

Notice that $\mathcal{R}$ is a Cohen-Macaulay ring and $\mathrm{a}(\mathcal{R}) = -1$.

Lemma 7.2 (cf. Ulrich)

$\mathrm{K}_{\mathcal{R}}(1) \cong J\mathcal{R}$ *as graded* $\mathcal{R}$*-modules.*

Corollary 7.3

Suppose that $\mathcal{R}$ *is a normal ring. Then* $J = Q : I$ *is integrally closed.*

Proof.

Since $\mathrm{K}_{\mathcal{R}}(1) \cong J\mathcal{R}$, $J\mathcal{R}$ is unmixed and of height one. Therefore $J\mathcal{R}$ is integrally closed in $\mathcal{R}$, whence J is integrally closed in R, because $\overline{J} \subseteq J\mathcal{R}$. □

The following is the key in our argument.

Theorem 7.4

The following conditions are equivalent.

(1) $\mathcal{R}$ *is a strongly almost Gorenstein graded ring*
i.e. $\exists$ *an exact sequence*

$$0 \to \mathcal{R} \to \mathrm{K}_{\mathcal{R}}(1) \to C \to 0$$

s.t. $MC = (\xi, \eta)C$ *for some homogeneous elements* $\xi, \eta \in M$.

(2) $\exists\ f \in \mathfrak{m},\ g \in I$, *and* $h \in J$ *s.t.*

$$IJ = gJ + Ih \quad \text{and} \quad \mathfrak{m}J = fJ + \mathfrak{m}h$$

When this is the case, $\mathcal{R}$ *is an almost Gorenstein graded ring.*

Theorem 7.5

Let $(R, \mathfrak{m})$ be a RLR with $\dim R = 2$, $|R/\mathfrak{m}| = \infty$. *Let* $\sqrt{I} = \mathfrak{m}$. *If* $I = \overline{I}$, *then* $\mathcal{R}(I)$ *is an almost Gorenstein graded ring.*

Proof.

Choose a parameter ideal Q s.t. $Q \subseteq I$ and $I^2 = QI$. Notice that I and $J = Q : I$ are integrally closed.

We choose elements $f \in \mathfrak{m}$, $g \in I$, and $h \in J$ s.t.

- f, h are a joint reduction of $\mathfrak{m}, J$
- g, h are a joint reduction of I, J

so that we have

$$\mathfrak{m}J = fJ + \mathfrak{m}h \quad \text{and} \quad IJ = gJ + Ih.$$

Hence $\mathcal{R} = \mathcal{R}(I)$ is an almost Gorenstein graded ring. □

Corollary 7.6

Let $(R, \mathfrak{m})$ be a RLR with $\dim R = 2$, $|R/\mathfrak{m}| = \infty$. Then $\mathcal{R}(\mathfrak{m}^\ell)$ is an almost Gorenstein graded ring for $\forall \ell > 0$.

For each ideal I of R, we set

$$\mathrm{o}(I) = \sup\{n \geq 0 \mid I \subseteq \mathfrak{m}^n\}.$$

Let R be a RLR with $\dim R = 2$ and $|R/\mathfrak{m}| = \infty$, $\sqrt{I} = \mathfrak{m}$. Then

$$I \text{ is a contracted ideal} \iff \mu_R(I) = \mathrm{o}(I) + 1.$$

Note that

$$I \text{ is integrally closed} \implies I \text{ is contracted and } I^2 = QI.$$

Theorem 7.7

Suppose that I is contracted and $\mathrm{o}(I) \leq 2$. Then $\mathcal{R}(I)$ is an almost Gorenstein graded ring.

For each ideal I of R, we set

$$\mathrm{o}(I) = \sup\{n \geq 0 \mid I \subseteq \mathfrak{m}^n\}.$$

Let R be a RLR with $\dim R = 2$ and $|R/\mathfrak{m}| = \infty$, $\sqrt{I} = \mathfrak{m}$. Then

$$I \text{ is a contracted ideal} \iff \mu_R(I) = \mathrm{o}(I) + 1.$$

Note that

$$I \text{ is integrally closed} \implies I \text{ is contracted and } I^2 = QI.$$

Theorem 7.7

Suppose that I is contracted and $\mathrm{o}(I) \leq 2$. Then $\mathcal{R}(I)$ is an almost Gorenstein graded ring.

Example 7.8

Let $R = k[[x, y]]$ be the formal power series ring over an infinite field k. We consider the ideals

$$I = (x^3, x^2y^3, xy^5, y^6) \quad \text{and} \quad Q = (x^3, y^6).$$

Then I is a contracted ideal of R with $I^2 = QI$ and $\mathrm{o}(I) = 3$, but $\mathcal{R}(I)$ is NOT an almost Gorenstein graded ring.

Theorem 7.9

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim R \geq 3$, $\exists \mathrm{K}_R$, $a_1, a_2, \ldots, a_r$ a subsystem of parameters for R $(r \geq 3)$. Set $Q = (a_1, a_2, \ldots, a_r)$. Then TFAE.

(1) *$\mathcal{R}(Q)$ is an almost Gorenstein graded ring.*

(2) *R is a RLR and $a_1, a_2, \ldots, a_r$ is a regular subsystem of parameters for R.*

Let $(R, \mathfrak{m})$ be a RLR with $d = \dim R \geq 2$ and $|R/\mathfrak{m}| = \infty$. Let Q be a parameter ideal of R s.t. $Q \neq \mathfrak{m}$ and set $I = Q : \mathfrak{m}$.

Theorem 7.10

Suppose that $d \geq 3$. Then TFAE.

(1) *$\mathcal{R}(I)$ is an almost Gorenstein graded ring.*

(2) *Either $I = \mathfrak{m}$, or $d = 3$ and $I = (x) + \mathfrak{m}^2$ for $\exists\, x \in \mathfrak{m} \setminus \mathfrak{m}^2$.*

Theorem 7.11

Suppose that $d = 2$. Then TFAE.

(1) *$\mathcal{R}(I)$ is an almost Gorenstein graded ring.*

(2) $\mathrm{o}(Q) \leq 2$.

Thank you so much for your attention.

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